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The paths of dislocations in wave pulses: an experimental test

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Abstract. Pulsed wavefields contain moving lines, called wave dislocations, where the amplitude is zero. As the lines move they sweep out surfaces called dislocation trajectories. The paper describes an experiment with ultrasound designed to test the theoretical prediction of Wright and Nye that, for small bandwidth, the trajectories are close to parts of frequency minimum surfaces: that is, surfaces on which the corresponding continuous-wave amplitude pattern has a minimum with respect to changes in frequency. 'Close' here means to second order in the bandwidth, and the prediction is indeed confirmed to this accuracy.

1. Background

When pulses of waves, originating from a common source, travel by different paths and interfere with one another they produce 'wave dislocations' (Nye and Berry 1974, Berry 1981, Nye 1981). These are generalisations of the interference fringes that would be produced if the waves were continuous (monochromatic) rather than pulsed. While perfect interference fringes, zeros of amplitude, made by continuous waves are lines stationary in space, the dislocations made by pulses are moving lines, which sweep out stationary surfaces called dislocation trajectories.

A receiver placed on a dislocation trajectory will typically find a signal like that of figure 1. The special feature is that the envelope of the oscillation has zero amplitude at a particular time, this being in fact the time at which the dislocation line moving through space passes through the receiver. The word 'dislocation' is used because the wave dislocation lines disrupt the spatial wave in the same way that crystal dislocation lines disrupt a crystal lattice.

For continuous waves the main features of a three-dimensional diffraction or interference pattern are well described simply by the arrangement of stationary line zeros or stationary dislocations. When pulses of waves interfere, the spatial pattern of amplitude of course changes with time; nevertheless its main features are still described by the array of dislocation lines, but in this case they are moving.

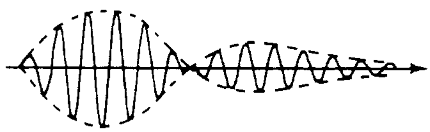

Figure 1. A pulse signal showing a dislocation. At the dislocation, seen in the centre of the diagram, the envelope of the signal has zero amplitude and the phase of the signal jumps by π.

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The question may now be asked: how are the moving dislocations of the pulsed wavefield related to the stationary dislocations of the continuous wavefield? An answer has been given (Wright and Nye 1982) for the case where the bandwidth of the original pulse is small, that is, where there is only a small departure from monochromaticity, as follows.

If the source oscillation were not a pulse but a single frequency ω at unit amplitude, the diffraction pattern would be specified in amplitude and phase by a complex transfer function $a(\boldsymbol{r}, \omega)$, where $\boldsymbol{r}$ is a position vector. Thus $a(\boldsymbol{r}, \omega)$ describes the ensemble of monochromatic diffraction patterns at various frequencies. Now suppose the source oscillation to be a pulse of narrow bandwidth and centre frequency ω_0. The theoretical prediction is that, to a good approximation, the trajectories of the moving dislocation lines will be parts of the surfaces described by

$$\left[\frac{\partial|a(\boldsymbol{r}, \omega)|}{\partial \omega}\right]_{\omega=\omega_0}=0, \qquad \left[\frac{\partial^2|a(\boldsymbol{r}, \omega)|}{\partial \omega^2}\right]_{\omega=\omega_0}>0. \tag{1}$$

In other words, the surfaces on which the amplitude of the continuous-wave diffraction pattern for frequency ω_0 is a minimum with respect to variations of frequency.

In this paper we describe an experiment with ultrasound designed to test relations (1).

2. Real and complex pulses

Although a dislocation may be defined, as we have seen, as the locus of points in space and time where the envelope of the physical signal is zero, strictly speaking, the envelope itself is not actually observed, only the oscillation within it. To take proper account of this the theory considers a complex source oscillation $\psi_0(t)$ and a corresponding complex received signal $\psi(\boldsymbol{r}, t)$, and it defines a dislocation as the locus of points in $(\boldsymbol{r}, t)$ where $\psi(\boldsymbol{r}, t)=0$.

The complex signals $\psi_0(t)$ and $\psi(\boldsymbol{r}, t)$ of the theory are related to the real signals measured in a physical experiment in the following way. Let $\psi_0(t)$ be written as

$$\psi_0(t)=f(t)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}, \tag{2}$$

where $f(t)$ is a slowly varying complex function and ω_0 is a chosen centre frequency within the bandwidth. Thus $f(t)$ represents a slow modulation, both in amplitude and phase, of a carrier wave of frequency ω_0. We could imagine two physical experiments: the first with Re $\psi_0(t)$ as the source signal giving Re $\psi(\boldsymbol{r}, t)$ at the receiver, and the second with Im $\psi_0(t)$ as the source signal giving Im $\psi(\boldsymbol{r}, t)$ at the receiver. Then the complex $\psi(\boldsymbol{r}, t)$ would be known. However, it can be shown (Walford *et al* 1977, Nye 1981) that, provided the modulation $f(t)$ is slow enough to have negligible frequency content outside the band $-\omega_0<\omega<\omega_0$, only one experiment, say with Re $\psi_0(t)$, is sufficient to determine both the real and the imaginary parts of $\psi(\boldsymbol{r}, t)$.

To do this first write $\psi(\boldsymbol{r}, t)$ at a given $\boldsymbol{r}$ as

$$\psi(t)=[X(t)+\mathrm{i}\,Y(t)]\,\mathrm{e}^{-\mathrm{i}\omega_0 t}, \tag{3}$$

where $X(t)$ and $Y(t)$ are real and slowly varying. Then multiply the observed signal

Re $\psi(t)$ by cos $\omega_0 t$ and sin $\omega_0 t$ to generate the two signals

$$\tfrac{1}{2}X(t)+\tfrac{1}{2}X(t)\cos 2\omega_0 t+\tfrac{1}{2}Y(t)\sin 2\omega_0 t$$

and

$$\tfrac{1}{2}Y(t)+\tfrac{1}{2}X(t)\sin 2\omega_0 t-\tfrac{1}{2}Y(t)\cos 2\omega_0 t.$$

Finally, apply a low-pass filter to remove frequencies outside the bandwidth of $X(t)$ and $Y(t)$. Thus we obtain two signals proportional to $X(t)$ and $Y(t)$, and the complex $\psi(t)$ is now known. A dislocation, defined by $\psi(t)=0$, can now be identified through (3) as a simultaneous zero of the two real functions $X(t)$ and $Y(t)$.

In our experiment Re $\psi(t)$ was multiplied by two square waves of frequency ω_0 in quadrature, rather than by pure sinusoids. The only difference is that higher harmonics of ω_0 are produced which are then removed by the low-pass filter.

3. Principle of the experiment

Our aim is to test relations (1), but instead of doing a series of continuous-wave experiments to establish $a(\boldsymbol{r}, \omega)$, and hence $|a(\boldsymbol{r}, \omega)|$, it suffices in fact to do a single experiment with a pulse, and to use Fourier analysis. Given that the source oscillation $\psi_0(t)$ produces the received signal $\psi(\boldsymbol{r}, t)$, it follows that, for given $\boldsymbol{r}$, by the definition of $a(\omega)$ as the transfer function,

$$a(\omega)=\bar{\psi}(\omega)/\bar{\psi}_0(\omega), \tag{4}$$

where bars denote Fourier transforms. Thus, by observing $\psi_0(t)$ and $\psi(t)$ and taking Fourier transforms we can deduce $a(\omega)$, and hence $|a(\omega)|$.

The experiment therefore consists of moving the receiver on to a dislocation trajectory (recognised as a simultaneous zero in $X(t)$ and $Y(t)$) and measuring $\psi_0(t)$ and $\psi(t)$. The modulus $|a(\omega)|$ is then deduced, and, if the prediction is correct, it should show a minimum near $\omega=\omega_0$, as in curve A of figure 2. There should be a special line on the trajectory, near a null line of the continuous-wave pattern for $\omega=\omega_0$, where the minimum is zero, as in curve B of figure 2.

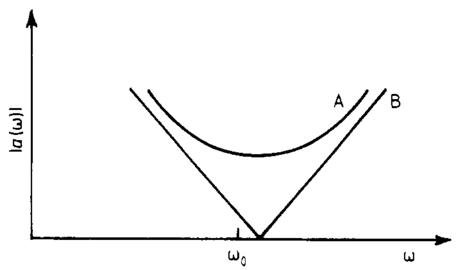

Figure 2. Theoretical predictions: the modulus of the transfer function $a(\omega)$ for a general point on a dislocation trajectory has a minimum near $\omega=\omega_0$, as in curve A; if the receiver is situated on a special line on the trajectory surface, the height of the minimum is zero, as in curve B.

4. The experiment

Even a single transducer, when stimulated by a pulse of oscillations, produces moving dislocation lines, in fact rings (Wright and Berry 1984), but experimentally it was more convenient to use two. The principle was as follows. Two sets of monochromatic uniform plane waves of equal amplitude at an angle to each other produce a set of parallel interference fringes; but the fringes in this instance are degenerate, being planes rather than lines. If, however, we introduce a linear modulation of amplitude, parallel to the wavefronts and in the same direction for both, as indicated in figure 3, the degeneracy is removed: the continuous-wave nulls are now a set of parallel lines. If, further, the sources of the plane waves are stimulated by a pulse rather than by a monochromatic oscillation (keeping the spatial modulation parallel to the wavefronts), the trajectories of (some of) the moving dislocations are parts of the parallel planes shown in figure 3 by the broken lines. This is the two-beam model worked out in detail by Wright and Nye (1982).

The experiment (figure 4) we now report approximates this ideal model by arranging two sources of ultrasonic pulses S_1, S_2 in air, about 15 cm apart and at an angle of 35° to one another. The centre frequency of 36 kHz corresponds to a wavelength of 9.5 mm. The two sources and the receiver were constructed from square piezoelectric bimorph devices (van Randeraat and Setterington 1974, § 6.4) approximately one wavelength

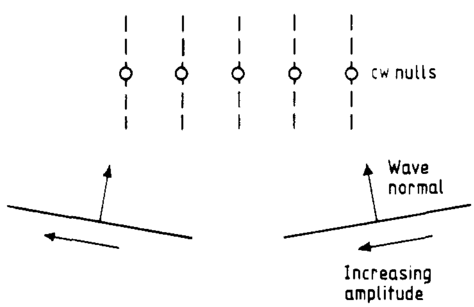

Figure 3. Two-beam model of Wright and Nye (1982). Two plane-wave pulses with amplitudes that increase linearly parallel to the wavefronts interfere to produce moving dislocation lines. The dislocation trajectories are vertical planes shown by broken lines.

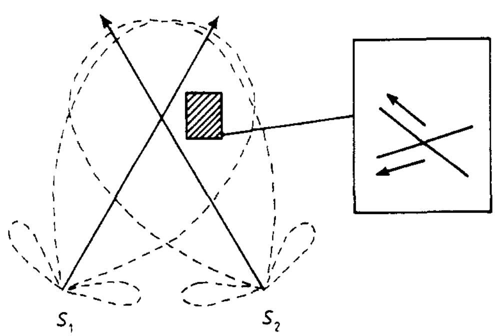

Figure 4. Approximate physical realisation of the ideal two-beam model.

across. We based the experiment loosely on figure 3 merely to make it easy to recognise the interference pattern; no precision was called for because the theory applies to any wavefield; indeed it was desirable to have some departure from the ideal arrangement to test the generality of the theory.

In the region shown shaded, which is not on the axis of either polar diagram, the two wave pulses have amplitudes that vary parallel to the wavefronts, both increasing in the same direction, as required. Experimentally it was easy to manipulate a dislocation trajectory into the shaded rectangle by moving one of the sources vertically, for the resulting phase change shifts the whole pattern, including the trajectories, sideways.

We must now consider the experimental interpretation of the transfer function $a(\boldsymbol{r}, \omega)$. In the first instance it represents the complex amplitude of the acoustic diffraction pattern physically present in space. However, as Wright and Nye (1982) point out, $a(\boldsymbol{r}, \omega)$ can equally well be defined as including the transfer functions of the receiving and transmitting apparatus, provided all processes are linear, without changing the theoretical results in any way. This is very convenient experimentally because, in general, the dislocations observed cannot be exactly the same as those physically present in the wavefield, an inevitable consequence of lack of perfection in the receiving transducer and its associated circuitry. For example, because of its finite size the receiver may fail to resolve a close pair of dislocations. This definition of $a(\boldsymbol{r}, \omega)$ avoids all problems of acoustic calibration.

Accordingly we now define $\psi_0(t)$ to be the electrical voltage that drives the two transmitting transducers. The envelope is a top-hat function to a very good approximation (figure 5) and gives a half-bandwidth σ of $0.042\omega_0$ to $0.046\omega_0$ depending on how

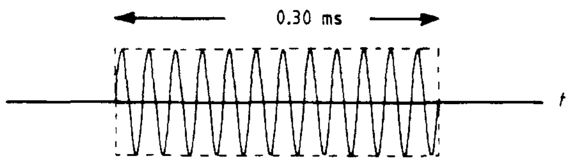

Figure 5. Source signal used in the experiment.

the bandwidth is defined. We shall take $\sigma = 0.044\omega_0$. The amplitude of the ultrasonic waves the transducers actually emit will not, of course, be represented by figure 5 because of their limited frequency response; in fact, as judged by the received signal, the wave amplitude rises smoothly until the driving pulse finishes (0.3 ms) and then decays exponentially, with a time constant of about 0.1 ms. Moreover, the two source transducers were slightly different. However, since their responses are already included in $a(\boldsymbol{r}, \omega)$ we need only know that they are linear.

(A possible objection to this definition of $\psi_0(t)$ is that the theory demands a slowly varying pulse envelope, which this certainly is not. However, since we are interested only in frequencies near ω_0, this form of pulse envelope is perfectly acceptable provided it has a spectrum near ω_0 that is the same, up to second derivatives, as some slowly varying pulse envelope, for example, a Gaussian: a condition that is easily satisfied.)

Coming now to the receiver, the ultrasonic transducer used for this purpose was followed by mixers and low-pass filters which extracted the two signals $X(t)$ and $Y(t)$. These were then amplified and displayed on an oscilloscope. The physical system described by $a(\boldsymbol{r}, \omega)$ was taken to be that which produces these signals; thus $a(\boldsymbol{r}, \omega)$

includes the frequency response not only of the receiver transducer but also of the associated circuitry leading to the observed voltages $X(t)$ and $Y(t)$. To detect a dislocation trajectory the receiver was moved until $X(t)$ and $Y(t)$, as observed on the oscilloscope, were simultaneously zero for some t, the arrival time of the dislocation.

Further processing of $X(t)$ and $Y(t)$, being outside the physical system, had to be done without distortion. To obtain them in digital form each of these slowly varying signals was sampled within a preset time window of 110 μs. The position of the time window was then moved and the process repeated to cover the duration (about 1 ms) of the whole signal. The main sources of random errors, namely draughts and unwanted echoes, had already been minimised by enclosing the experimental chamber with acoustically absorbing material. Remaining random errors were reduced by averaging over 10 to 50 separate recordings at each observing station.

These data were filtered using a Hamming window (see, for example, Blackman and Tukey 1958) and then Fourier analysed (using the NAG library routine CO6ECF). The transfer function $a(\omega)$ of the system was then calculated by complex division of the Fourier transform $\bar{\psi}(\omega)$ of the received signal by the spectrum $\bar{\psi}_0(\omega)$ of the original pulse. For this purpose $\bar{\psi}_0(\omega)$ was found analytically by taking $\psi_0(t)$ to be a harmonic waveform modulated by a rectangular time window (figure 5).

The complex transfer functions at three stations along one trajectory are shown in Argand diagrams in figure 6(a), the curves being parametrised by frequency, and the corresponding moduli $|a(\omega)|$ are shown in figure 6(b). The range of frequencies illustrated in figure 6(a) is narrower than that in figure 6(b), and is only the central part of the total bandwidth 2σ. The moduli $|a(\omega)|$ at the three stations show minima very close to $\omega = \omega_0$, and to this extent the theory is verified. The minimum in the second diagram of figure 6(b) is very nearly zero, approximating curve B in figure 2. The different trajectories were 1-2 cm apart and the precision of setting the receiver on the selected trajectory was 0.1 mm. Moving the receiver away from the trajectory, rather than along it, would have caused the minima in figure 6(b) to shift sideways, rather than to change their height. Nicholls (1984) gives further experimental details.

The departure of the minima from $\omega = \omega_0$ is $0.09\sigma = 0.004\omega_0$, where σ is the half-bandwidth of the source signal. It is too large to be accounted for by experimental errors and can be explained as due to the approximate nature of the perturbation

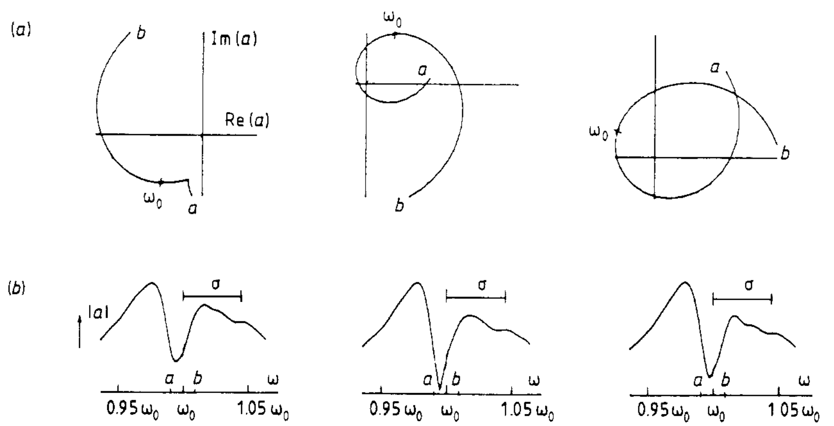

Figure 6. (a) Argand diagrams of the measured transfer function $a(\omega)$ for three rceiver points on a chosen dislocation trajectory. (b) The modulus of the transfer functions $a(\omega)$ shown in (a) plotted against ω. The letters correspond with those in (a).

theory; relations (1) are only exact in the limit of small bandwidth. Theoretically the minimum of $|a(\omega)|$ should occur at a frequency differing from ω_0 by an amount $(\Delta\omega)_{\text{th}}$ that is of second order in the bandwidth. An order-of-magnitude estimate, derived in the appendix from the Wright-Nye theory, is $(\Delta\omega)_{\text{th}} \sim \frac{1}{2}\sigma^2 \Delta\tau_0$, where $2\Delta\tau_0$ is the relative delay time between the two pulses arriving at the receiver when it is close to the null line of the continuous-wave pattern for $\omega = \omega_0$. The possible values of $2\Delta\tau_0$ are $\frac{1}{2}T, \frac{3}{2}T, \ldots$, where T is the period $(2\pi/\omega_0)$, and we estimate $2\Delta\tau_0 = \frac{1}{2}T$ or possibly $\frac{3}{2}T$, since the order of interference in the experiment was arranged to be low. Hence

$$(\Delta\omega)_{\text{th}} \sim \frac{\pi\sigma^2}{4\omega_0} \quad \text{or} \quad \frac{3\pi\sigma^2}{4\omega_0}$$

and, with $\sigma = 0.044\omega_0$, this gives $(\Delta\omega)_{\text{th}} \sim 0.0015\omega_0$ or possibly $0.0046\omega_0$. In view of the uncertainty of a numerical factor, assumed to be of order 1, either value is compatible with the measured $0.004\omega_0$. Thus, the small observed frequency discrepancy is about what ought to be expected from the known finite bandwidth of the system.

5. Further discussion

There is one feature of these results that needs additional explanation. Wright and Nye (1982, p 372) show that their two-beam model is degenerate, in the sense that the dislocation trajectories obey relations (1) exactly rather than approximately, and that the trajectories pass exactly through the continuous-wave null lines for frequency ω_0. A small extension of their argument shows that this degenerate feature is possessed by all models where the transmitted pulse given by (2) has a real envelope function $f(t)$, and where the receiver picks up just two faithful versions of this pulse that differ only in amplitude and arrival time. The spatial arrangement of the continuous-wave fringes is not important. If this description fitted our experiment, we should expect $(\Delta\omega)_{\text{th}} = 0$, contrary to observation.

In fact our arrangement departs significantly from that just described because of the response characteristics of the two transmitting transducers. They were not resonant at exactly the frequency of their excitation, ω_0, and therefore the acoustic pulses they produced were not simply amplitude-modulated harmonic waves of frequency ω_0 ($f(t)$ real), but showed a phase modulation as well ($f(t)$ complex). Loosely speaking, the frequency was slightly different in the rising and falling parts of the pulses. This would be sufficient by itself to remove the degeneracy, but in addition the resonant frequencies of the two transmitting transducers were slightly different. The net effect is that the ideal two-pulse degeneracy was not present.

To summarise, we have inquired into the frequency response of the transmitting transducers just to satisfy ourselves that the system observed had sufficient generality. Having done so, we need not take account of the transmitter response in detail because it is already included within the transfer function of the physical system under observation.

Acknowledgments

The ultrasonic apparatus we used was built by Dr M E R Walford and some of the computer programs were prepared by Dr M F L Harper. We are grateful to them, and also to Dr G P Davidson and Dr V Humphrey who designed and built other parts of the equipment.

Appendix

We derive here the formula $(\Delta\omega)_{\text{th}} \sim \frac{1}{2}\sigma^2 \Delta\tau_0$ used at the end of § 4. Wright and Nye (1982) give in their § 3.8 a theory for small bandwidth valid in the immediate neighbourhood of a continuous-wave null line, and we assume this to be applicable, at least for the minimum in the central lower diagram of figure 6, where the receiver was closest to the null.

The theory uses a real-imaginary representation of the complex transfer function $a(\boldsymbol{r}, \omega)$, thus

$$a(\boldsymbol{r}, \omega) \equiv \mathrm{e}^{\mathrm{i}\beta_0}[P(\boldsymbol{r}, \omega) + \mathrm{i}Q(\boldsymbol{r}, \omega)], \tag{A1}$$

where β_0 is an arbitrary phase chosen to make $(\partial P/\partial\omega)_{\omega=\omega_0}$ vanish at a chosen origin $\boldsymbol{r}=0$ on the null line. The required quantity $(\Delta\omega)_{\text{th}}$ is given (on p 369 after equation (87)) by the formula

$$(\Delta\omega)_{\text{th}} = \sigma^2 b_8 / b_4 \tag{A2}$$

in terms of two coefficients b_4 and b_8 which are proportional to frequency derivatives of $Q(\boldsymbol{r}, \omega)$ taken at $\omega = \omega_0$ and $\boldsymbol{r} = 0$:

$$b_4 = \left(\frac{\partial Q}{\partial \omega}\right)_{\substack{\omega=\omega_0\\ \boldsymbol{r}=0}}, \qquad b_8 = \frac{1}{2}\left(\frac{\partial^2 Q}{\partial \omega^2}\right)_{\substack{\omega=\omega_0\\ \boldsymbol{r}=0}}. \tag{A3}$$

Our task is to estimate the ratio b_8/b_4, which has the physical dimensions of time. The essential question is: what determines this timescale? First of all, notice that it is entirely to do with the properties of the transfer function $a(0, \omega)$, which measures how the continuous-wave diffraction amplitude depends on frequency at $\boldsymbol{r} = 0$, and therefore it cannot involve σ, which is a property of the pulse.

The only other timescales in the problem are set by the centre frequency ω_0 and by $\Delta\tau_0$, which is half the relative delay time between the two waves arriving at $\boldsymbol{r} = 0$. In the ideal two-beam model of Wright and Nye, from their equation (90),

$$a(0, \omega) \equiv a_1(\omega \Delta\tau_0), \tag{A4}$$

where a_1 denotes a (dimensionless) function of a single dimensionless variable. That is, the timescale for ω is set by $\Delta\tau_0$. This result is general. The role of ω_0 is solely to decide the position of the continuous-wave null line in question.

It follows from equations (A1) and (A4) that

$$Q(0, \omega) \equiv Q_1(\omega \Delta\tau_0)$$

where Q_1 likewise denotes a (dimensionless) function of a single dimensionless variable. Then, by differentiation and use of (A3),

$$\frac{b_8}{b_4} = \tfrac{1}{2}\Delta\tau_0 \frac{Q_1''(\omega_0 \Delta\tau_0)}{Q_1'(\omega_0 \Delta\tau_0)}.$$

The factor $Q_1''(\omega_0\Delta\tau_0)/Q_1'(\omega_0\Delta\tau_0)$ is dimensionless and we assume it to be of order 1. It follows from (A2) that

$$(\Delta\omega)_{\rm th} \sim \tfrac{1}{2}\sigma^2\Delta\tau_0.$$

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